COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **13** (4) (1993) 421–433

HOMOMORPHISMS TO ORIENTED CYCLES

PAVOL HELL, HUISHAN ZHOU and XUDING ZHU

Received February 6, 1990
Revised June 25, 1991

We discuss the existence of homomorphisms to oriented cycles and give, for a special class of cycles C, a characterization of those digraphs that admit a homomorphism to C. Our result can be used to prove the multiplicativity of a certain class of oriented cycles (and thus complete the characterization of multiplicative oriented cycles), as well as to prove the membership of the corresponding decision problem in the class $NP \cap \mathrm{co}NP$. We also mention a conjecture on the existence of homomorphisms to arbitrary oriented cycles.

1. Introduction

The problem of existence of graph homomorphisms has attracted considerable attention, [1], [2], [3], [4], [10], [13], [14], [15], [20], [21], [25]. From an algorithmic point of view, the problem is known to be NP-complete when the target is a fixed undirected graph G of chromatic number greater than two, and polynomial for all other undirected graphs, [15]. No such clear distinction is known for digraphs, [2], [3], [4], [20], although some conjectures for special cases have been proposed, [1]. One result of note, [8], is a polynomial algorithm for the existence of a homomorphism to an oriented path. However, the existence of homomorphisms to oriented cycles appears to be a harder problem and no general polynomial algorithm is known at this time.

Another recent source of interest in the same problem is Hedetniemi's conjecture [5], [6], [12], [22], [24], which states that the chromatic number of the product of two n-chromatic graphs is n. (The product here is the conjunction [11], also known as the categorical product [22], in which (a,b) is adjacent to (c,d) just if a is adjacent to c and b to d.) This led to the definition of a *multiplicative* (directed or undirected) graph W, [9], as one for which graphs non-homomorphic to W are closed under taking products (see also [23]). In other words, W is multiplicative just if $G \not\to W$ and $G' \not\to W$ implies that $G \times G' \not\to W$. It is easy to see that Hedetniemi's conjecture asserts that complete graphs are multiplicative. Some multiplicative graphs and digraphs were given in [9], [26], [27]. Again, multiplicative oriented paths have been completely characterized, [26], while the situation for oriented cycles is more difficult. In particular, [26] introduced a special class

AMS subject classification code (1991): 05 C 20, 05 C 99

of cycles, called $\mathscr{C}$-cycles, and showed that among oriented cycles only the $\mathscr{C}$-cycles could be multiplicative. (A simpler proof of this result is given in [27].) However, the problem of whether or not all $\mathscr{C}$-cycles were multiplicative, remained open.

We introduce a more general class of oriented cycles, called $\mathscr{B}$-cycles, and give a characterization of those digraphs which are homomorphic to a fixed $\mathscr{B}$-cycle. This result will allow us to prove (in a subsequent paper [16]) that all $\mathscr{C}$-cycles are indeed multiplicative, thus completing the characterization of multiplicative oriented cycles. It will also follow from our result that the existence problem for homomorphisms to a fixed $\mathscr{B}$-cycle is in $NP\cap coNP$. We shall mention corresponding results about homomorphisms to a fixed oriented path, and a possible extension of our main result to any oriented cycle of non-zero length. Proving this extension would verify that the existence problem for homomorphisms to any oriented cycle of non-zero length is in $NP\cap coNP$, and possibly even suggest a polynomial algorithm for it.

A *homomorphism* of a digraph G to a digraph H is a mapping of the vertex sets $V(G) \to V(H)$ which preserves the edges, i.e., such that $xy \in E(G)$ implies $f(x)f(y) \in E(H)$. If such a homomorphism exists, we say G is *homomorphic* to H and write $G \to H$. Otherwise we write $G \not\to H$.

An *oriented path* P is a digraph obtained from an undirected path by orienting its edges and assigning to it a positive direction. Thus an oriented path P is a digraph given by its sequence of vertices $\langle p_0, p_1, \ldots, p_n \rangle$, such that, for each $i \in \{0, 1, \ldots, n-1\}$, either $p_i p_{i+1} \in E(P)$ (a *forward* edge of P), or $p_{i+1} p_i \in E(P)$ (a *backward* edge of P), and such that P has no other edges. The direction of P is emphasized by saying that p_0 is the *initial point*, $i(P)$, of P, and p_n the *terminal point*, $t(P)$, of P, respectively. Expressions such as "u precedes (or follows) v on P", or "z is between x and y on P", also refer to this order on P. Changing the direction of P results in the path $P^T = \langle p_n, p_{n-1}, \ldots, p_0 \rangle$. Note that P^T is the same digraph as P, only traversed in the opposite order. If $P = \langle p_0, p_1, \ldots, p_n \rangle$ and $P' = \langle p'_0, p'_1, \ldots, p'_m \rangle$ are oriented paths with disjoint vertex-sets, the *concatenation* of P and P' is the oriented path $P \circ P' = \langle p_0, p_1, \ldots, p_n = p'_0, p'_1, \ldots, p'_m \rangle$. We often concatenate given paths with the special oriented path $A = \langle a, a' \rangle$, consisting of a single forward edge aa'.

Let $P = \langle p_0, p_1, \ldots, p_n \rangle$ be an oriented path and let $u = p_i$ precede $v = p_j$ in P (i.e., let $i < j$). The *interval* of P from u to v is the oriented path $P[u,v] = \langle p_i, p_i + 1, \ldots, p_j \rangle$ and the *interval* of P from v to u is the oriented path $P[v,u] = P[u,v]^T$. We also let $P[.,v] = P[i(P),v]$, and $P[u,.] = P[u,t(P)]$.

Let P be an oriented path. The *length* $l(P)$ of P is the number of forward edges of P minus the number of backward edges of P. We say that P is *minimal* if it contains no proper interval of the same length. An interval I of P is called minimal if I is a minimal path. The *distance* from u to v in P is $d_P(u,v) = l(P[u,v])$. The *level* of u in P is $\lambda_P(u) = l(P[.,u])$. Note that homomorphisms of paths preserve distance, i.e., if $f: P \to P'$ is a homomorphism and $u, v \in P$ then $d_{P'}(f(u), f(v)) = d_P(u,v)$.

A *directed path* (*interval*) is an oriented path (interval) with all edges in the same direction. If they are forward edges it is called a *forward directed path* (*interval*), otherwise it is called a *backward directed path* (*interval*).

An *oriented cycle* C is a digraph obtained from an undirected cycle by orienting its edges and assigning to it a positive direction. Thus an oriented cycle C is a digraph given by its circular sequence of vertices $\langle c_0, c_1, \ldots, c_n, c_0 \rangle$, such that, for each $i \in \{0,1,\ldots,n\}$, either $c_i c_{i+1} \in E(C)$ (a *forward* edge of C), or $c_{i+1}c_i \in E(C)$ (a *backward* edge of C), and such that C has no other edges. (Subscript addition modulo n.) Since we do not distinguish an initial vertex of an oriented cycle, $\langle c_0, c_1, \ldots, c_n, c_0 \rangle = \langle c_1, c_2, \ldots, c_n, c_0, c_1 \rangle$, we usually choose a most convenient vertex to start listing C. Note that we can view an oriented cycle as an oriented path in which the initial and terminal vertices have been identified, and in this spirit we shall use some of the definitions given for oriented paths also for oriented cycles. In particular, the length of the oriented cycle C is the difference between the number of forward edges and the number of backward edges of C; an interval of C is an interval of $\langle c_0, c_1 \ldots, c_n \rangle$, where $\langle c_0, c_1, \ldots, c_n, c_0 \rangle$ is any of the different ways of listing C.

Let $P = \langle p_0, p_1, \ldots, p_m \rangle$ be an oriented path and $C = \langle c_0, c_1, \ldots, c_n \rangle$ an oriented cycle. Consider a homomorphism $f : P \to C$ such that $f(p_i) = f(p_{i+2})$ for some i. Define $P' = \langle p_0, p_1, \ldots, p_i, p_{i+3}, \ldots, p_m \rangle$, and define $f'(p_j) = f(p_j)$ for $j = 0, 1, \ldots, i, i+3, \ldots, m$. Note that $f' : P' \to C$ is a homomorphism; we shall say that it is obtained from $f : P \to C$ by a *simplification step*. If $f'' : P'' \to C$ is obtained from $f : P \to C$ by a sequence of simplification steps, we shall say that $f : P \to C$ *simplifies* to $f'' : P'' \to C$. We shall say that the homomorphism $f : P \to C$ *wraps* P around C if f simplifies to $f'' : P'' \to C$ where $P'' = \langle p''_0, p''_1, \ldots, p''_{n+1} \rangle$ and $f''(p''_j) = c_j$, for $j = 0, 1, \ldots, n$, and $f''(p''_{n+1}) = c_0$. We shall say that the homomorphism $f : P \to C$ *winds* P around C if some restriction of f to an interval of P wraps the interval around C. Finally, we shall say that the oriented path P *can* (*can not*) *be wound around* C if there is (isn't) a homomorphism $f : P \to C$ which winds P around C.

Definition 1.1. An oriented cycle $C = \langle c_0, c_1, \ldots, c_n, c_{n+1}, \ldots, c_m, c_0 \rangle$, where $n \geq 2$, is a $\mathscr{B}$*-cycle* (*with parameter* n), if $\langle c_0, c_1, \ldots, c_n \rangle$ is a forward directed path (of length n), and $\langle c_0, c_m, c_{m-1}, \ldots, c_{n+1}, c_n \rangle$ an oriented path of length $n-1$ which does not contain an interval of length n.

Note that C has length 1. Note further that $I = \langle c_0, c_1, \ldots, c_n \rangle$ is the only minimal interval of C of length n, i.e., every interval of C is of length at most n, and every interval of length n must contain I. Therefore, if P is any minimal oriented path of length n and if $h : P \to C$ is a homomorphism, then $h(i(P)) = c_0$ and $h(t(P)) = c_n$. Furthermore, if P is an oriented path which contains an interval of length greater than n and if $h : P \to C$ is a homomorphism, then h must wind P around C.

We now give our main result, a characterization of the class of digraphs which are homomorphic to a fixed $\mathscr{B}$-cycle.

Theorem 1.2. *Let C be a $\mathscr{B}$-cycle and G any digraph. Then $G \not\to C$ if and only if there exists an oriented path P such that $P \to G$ but $P \not\to C$.*

If $G \to C$ and P is an oriented path such that $P \to G$, then of course $P \to C$ by composition. Thus the sufficiency of the condition is obvious. The remainder of this paper consists of proving the necessity. Thus we shall prove that G is homomorphic to C provided all paths homomorphic to G are also homomorphic to C.

Note that a homomorphism f of a path $P = \langle p_0, p_1, \dots, p_m \rangle$ to G may be viewed as a walk in G, simply by identifying it with the sequence of vertices $f(p_0), f(p_1), \dots, f(p_m)$. We could also call a *walk pattern of* G any path P which is homomorphic to G. In this terminology, our main theorem would assert that G is homomorphic to C if and only if each walk pattern of G is homomorphic to C. Since this terminology is somewhat unusual, we shall avoid it in the sequel. However, it may help the reader to bear this point of view in mind when reading the proofs. In particular, we frequently define paths $P = \langle p_0, p_1, \dots, p_m \rangle$ and homomorphisms $f: P \to G$, having first in mind the walk $f(p_0), f(p_1), \dots, f(p_m)$ in G.

2. The mapping ψ

From now on we assume that $C = \langle c_0, c_1, \dots, c_n, c_{n+1}, \dots, c_m, c_0 \rangle$ is a fixed $\mathcal{B}$-cycle with parameter n, and that G is a fixed digraph such that every path P homomorphic to G is also homomorphic to C. We proceed to construct a homomorphism $G \to C$.

First we associate with C a path

$$R = \langle r_0, r_1, \dots, r_n, r_{n+1}, \dots, r_m, r_{m+n}, r_{m+n-1}, \dots, r_{m+1} \rangle,$$

such that $r_i r_j$ is a forward (respectively backward) edge just if $c_i c_j$ is a forward (respectively backward) edge of C, where we let $c_{m+n-i} = c_i$, $i = 0, 1, \dots, n-1$. Note that there is a natural homomorphism $R \to C$, taking r_i to c_i. For $v \in R$, $Ind(v)$ denotes the index of v, i.e., $Ind(v) = i$ just of $v = r_i$. We write $u \leq v$ (or $u < v$) if $Ind(u) \leq Ind(v)$ (respectively $Ind(u) < Ind(v)$) on R; thus $r_i \leq r_j$ just if $i \leq j$. When we speak of the maximum of a set of vertices of R, we are referring to this order. (Note that this is not the ordering imposed by R, in which r_m is followed by r_{m+n}, then r_{m+n-1}, etc.)

It follows from the definition of R that $D = \langle r_0, r_1, \dots, r_n \rangle$ is a directed interval of R of length n. It also follows that for every vertex x of R we have $0 \leq \lambda_R(x) \leq n$, and $\lambda_R(x) = 0$ if and only if $x = r_0$. Thus D is the only minimal interval of R of length n. Hence if P is any minimal oriented path of length n and if $h: P \to R$ is any homomorphism, then $h(i(P)) = r_0$ and $h(t(P)) = r_n$.

We shall denote by $\mathcal{P}$ the set of all paths P homomorphic to G such that $0 < \lambda_P(x) \leq n$ holds for all vertices of P except for $i(P)$. Each interval of any $P \in \mathcal{P}$ has length at most n. It is well known, cf. [23], [9], that this implies that P is homomorphic to D, and hence also homomorphic to R.

Definition 2.1. Define $\phi: \mathcal{P} \to V(R)$ as follows: For $P \in \mathcal{P}$,

$$\phi(P) = \max\{h(t(P)) : \ h : P \to R\}.$$

Define $\psi: V(G) \to V(R)$ by

$$\psi(x) = \min\{\phi(P) : \ P \in \mathcal{P}, \text{ and for some } h : P \to G, \ h(t(P)) = x\}.$$

Since there is a natural homomorphism from R to C, ψ induces a mapping of G to C. In this section we show that this induced mapping has some nice properties.

However it is not, in general, a homomorphism. In the next section, we will use ψ to construct a true homomorphism of G to C.

Definition 2.2. Put

$$K = \{x \in V(G) : \ \psi(x) = r_{m+n}\}$$
$$K_1 = \{x \in V(G) : \ r_1 \le \psi(x) \le r_m\} \quad \text{and}$$
$$K_2 = \{x \in V(G) : \ r_{m+1} \le \psi(x) \le r_{m+n-1}\}.$$

Put $L = K_1 \cup K_2$.

It follows from the definition of $P \in \mathcal{P}$ (and from the fact that homomorphisms preserve distances), that $\phi(P) \neq r_0$; whence each $\psi(x) \neq r_0$ and $V(G) = K \cup L$.

Lemma 2.3. *For each vertex $x \in K_2$, there exists a path $P \in \mathcal{P}$ and homomorphisms $g: P \to G$, $h: P \to R$ such that $g(t(P)) = x$, $h(i(P)) = r_{m+n}$, and $h(t(P)) = \psi(x) = \phi(P)$. In particular, P contains no interval of length n.*

Proof. The definition of $\psi(x)$ implies that there is a path P' and homomorphisms $g' : P' \to G$, $h' : P' \to R$ such that $g'(t(P')) = x$ and $h'(t(P')) = \psi(x) = \phi(P')$. Since $\langle r_{m+n}, r_{m+n-1}, \dots, r_{m+1}\rangle$ is a directed path and $\psi(x) \geq r_{m+1}$, it follows from the definition of $\phi(P')$ that h' maps some vertex of P' to r_{m+n}. Let v be the last vertex of P' such that $h'(v) = r_{m+n}$. Let $P = P'[v, .]$, and let g, h be the corresponding restrictions of g', h'. It is now easy to see that the conclusions hold. ∎

Remark. The situation is different for vertices $x \in K_1$. In fact, any P with homomorphisms $g : P \to G$, $h : P \to R$ such that $g(t(P)) = x$ and $h(t(P)) = \phi(P) = \psi(x)$ does contain an interval of length n. Indeed, if $\lambda_P(v) < n$ for all v then P maps to $\langle r_{m+n}, r_{m+n-1}, \dots, r_{m+1}\rangle$ contradicting the fact that $\phi(P) \leq r_m$. Let v be the first vertex on P of level n, and let f be any homomorphism of P to R (respectively to C). Since $P[., v]$ is a minimal interval of length n, we have $f(i(P)) = r_0$ (respectively $f(i(P)) = c_0$) and $f(v) = r_n$ (respectively $f(v) = c_n$). This implies in particular that the length of P is determined by x, namely $l(P) = \lambda_R(\psi(x))$.

For $x \in L$, let $\mathcal{P}_x$ denote the set of all paths $P \in \mathcal{P}$ which admit homomorphisms $g: P \to G$ and $h: P \to R$ such that $g(t(P)) = x$, $h(t(P)) = \phi(P) = \psi(x)$, and $h(i(P)) = r_{m+n}$ if $x \in K_2$ or $h(i(P)) = r_0$ if $x \in K_1$. (According to the last remark, $h(i(P)) = r_0$ is automatic for $x \in K_1$.) The above remark also implies that the length of all $P \in \mathcal{P}_x$ is the same for $x \in K_1$; a similar argument shows the same for $x \in K_2$.

Lemma 2.4. *Let $x \in L$ and let $P \in \mathcal{P}_x$. Then P can not be wound around C.*

Proof. Suppose $P \in \mathcal{P}_x$ and $f: P \to C$ is a homomorphism which winds P around C.

Assume first that $x \in K_2$: Note that $f(i(P)) \neq c_0$ since P does not contain an interval of length n. Thus $f(v) \neq c_0$ for all $v \in P$, since the distance from $f(i(P))$ to c_0 along any direction of C is non-positive and the distance from $i(P)$ to any other point of P is positive. Therefore f does not wind P around C.

Assume now that $x \in K_1$: Then $f(i(P)) = c_0$, according to the remark. Since all $\lambda_P(v) > 0$ for $v \neq i(P)$, f must map P around C in the positive direction. Since f winds P around C, some vertex $v \neq i(P)$ of P has $f(v) = c_0$. Then $l_P(v) = 1$ and so $P' = P[v, .]$ contains no interval of length n. Furthermore P' contains no vertex of negative level. Therefore there is a homomorphism $h: P' \to R[r_{m+n}, r_{m+1}]$ such that $h(v) = r_{m+n}$. We may view f restricted to $P[.,v]$ as a homomorphism to R, with $f(v) = r_{m+n}$. This restriction of f, together with the homomorphism h then yield a homomorphism $g: P \to R$, such that $g(t(P)) \geq r_{m+1}$. This contradicts the assumption that $\phi(P) = \psi(x) \leq r_m$. ∎

Lemma 2.5. *Let $x, y \in L$ and $xy \in E(G)$. Then any $P_x \in \mathcal{P}_x$ has length less than n, and any $P_y \in \mathcal{P}_y$ length more than 1.*

Proof. Suppose $P_x \in \mathcal{P}_x$ has length n, and let $P' = P_x \circ A$. (Recall that A is the path consisting of a single forward edge aa'.) Then P' has length $n+1$ and is homomorphic to G (any homomorphism $P_x \to G$ which takes $t(P_x)$ to x may be extended to $P' \to G$ by mapping a' to y). According to our assumption, it is also homomorphic to C. Any homomorphism $P' \to C$ must wind P' around C, in order to achieve the length $n+1$. In fact, even its restriction to P_x must wind P_x around C, for the length of C is 1. Since this contradicts Lemma 2.4, we have $l(P_x) < n$.

Suppose $P_y \in \mathcal{P}_y$ has length 1, and let $P'' = P_x \circ A \circ P_y^T$. Then the length of $P''[a,.]$ is zero, and hence for each $u \in P_y^T \setminus i(P_y)$ in P'' the distance from a, being the same as the distance from $i(P_y)$, is positive. We first show that there is no $u \in P_y^T \setminus i(P_y)$ for which $\lambda_{P_y}(u) = n$. Suppose there is; then we have $\lambda_{P''}(u) \geq n+1$. Thus any homomorphism $P'' \to C$ must wind P'' around C. On the other hand, there exists such a homomorphism $f: P'' \to C$, because P'' is obviously homomorphic to G (take a to x and a' to y). Since $i(P'')$ is the initial vertex of a minimal interval of P'' of length n, we have $f(i(P'')) = c_0$. Let $v \in P''$ be the first vertex of P'' after $i(P'')$ such that $f(v) = c_0$. Then $v \in P_y$, for otherwise f would wind P_x around C contrary to Lemma 2.4. Also v must precede u in P'' because $\lambda_{P''}(u) \geq n+1$. Therefore $v \neq i(P_y)$. This is a contradiction because it implies that $d_{f(P'')}(f(a), f(v)) \leq 0$ while $d_{P''}(a,v) > 0$.

Therefore all $\lambda_{P_y}(u) \leq n-1$, and hence $y \in K_2$. Then $l(P_y) = 1$ implies that $\psi(y) = \phi(P_y) = r_{m+n-1}$. Let, as above, $P' = P_x \circ A$. Then $P' \in \mathcal{P}$ because $l(P_x) \leq n-1$. There is a homomorphism of P' to G which takes a' to y. Thus we have $\phi(P') \geq \psi(y) \geq r_{m+n-1}$. Let $h: P' \to R$ be a homomorphism such that $h(t(P')) \geq r_{m+n-1}$. Now $h(t(P')) = h(a') \neq r_{m+n}$, because $h(a')$ is the end of the edge starting in $h(a)$, while r_{m+n} has indegree zero. Also $h(t(P')) = h(a') \neq r_{m+n-1}$, otherwise $h(a) = r_{m+n}$ which contradicts the assumption that $\phi(P_x) = \psi(x) \leq r_{m+n-1}$. This final contradiction proves the lemma. ∎

Corollary 2.6. *Assume $x, y \in L$, $xy \in E(G)$. If $P_x \in \mathcal{P}_x$, $P_y \in \mathcal{P}_y$ then $P_x \circ A \in \mathcal{P}$, $P_y \circ A^T \in \mathcal{P}$.*

Lemma 2.7. *Assume $x, y \in K_1$. If $xy \in E(G)$, then $\psi(x)\psi(y) \in E(R)$.*

Proof. Suppose first that $\psi(x) < \psi(y)$. Let $P_x \in \mathcal{P}_x$ and let $P' = P_x \circ A$. There is a homomorphism $g: P' \to G$ with $g(a) = x$ and $g(a') = y$. Since $P' \in \mathcal{P}$ (by the above Corollary) and $g(t(P')) = y$, there is a homomorphism $h': P' \to R$ such that $h'(a') \geq \psi(y)$. On the other hand, $h'(a) \leq \phi(P_x) = \psi(x) < \psi(y)$, since h' restricted to P_x is a homomorphism. Now $h'(a)h'(a') \in E(R)$, and so $Ind(h'(a')) \leq Ind(h'(a))+1$. (Since $\psi(x) < \psi(y)$, $\psi(x)$ can not be r_m.) Then $Ind(h'(a)) \leq Ind(\psi(x)) < Ind(\psi(y)) \leq$

$Ind(h'(a')) \leq Ind(h'(a))+1$ implies that $h'(a)=\psi(x)$, $h'(a')=\psi(y)$ and therefore $\psi(x)\psi(y)\in E(R)$.

A similar argument applies in the case $\psi(x) > \psi(y)$. One only needs to use $P_y \in \mathcal{P}_y$ and $P''=P_y \circ A^T$, and a homomorphism $h'': P'' \to R$ such that $h''(a') \geq \psi(x)$.

It remains to consider the case $\psi(x)=\psi(y)$. Let P', h', P'' and h'' be defined as above. Let $\psi(x)=\psi(y)=r_i$. Then as above, $h'(a)\leq\psi(x)=r_i=\psi(y)\leq h'(a')$. Now $h'(a)h'(a')\in E(R)$ implies that $Ind(h'(a))\geq Ind(h'(a'))-1\geq i-1$. Therefore either $h'(a)=r_i$, or $h'(a)=r_{i-1}$. But $h'(a)$ cannot be r_{i-1}, because the homomorphism inherent in the definition of $\phi(P_x)$ maps P_x to a path that starts at r_0 and ends at r_i, so (as homomorphisms preserve distances) h' cannot map P_x to a path that starts at r_0 and ends at r_{i-1}. (By the remark, $h'(i(P_x))=r_0$). Therefore $h'(a)=\psi(x)=r_i$ and $h'(a')=r_{i+1}$ (or r_{m+n} if $i=m$). So r_ir_{i+1} (or r_mr_{m+n})$\in E(R)$. The same argument applied to P'' and h'' will show that $r_{i+1}r_i$ (or $r_{m+n}r_m$)$\in E(R)$. This is a contradiction because R has no pair of opposite edges. Therefore this case can not happen and the lemma is proved. ∎

Lemma 2.8. *Assume $x,y\in K_2$. If $xy\in E(G)$, then $\psi(x)\psi(y)\in E(R)$.*

Proof. Again, we take $P_x \in \mathcal{P}_x$, $P_y \in \mathcal{P}_y$, $P' = P_x \circ A$, and $P'' = P_y \circ A^T$; note that $P' \to G$ and $P'' \to G$ with a going to x and a' to y. Since $P' \in \mathcal{P}$, there exists a homomorphism $h' : P' \to R$ with $h'(a') \geq \psi(y)$. Clearly, $h'(a) \leq \psi(x)$. As there is an edge from $h'(a)$ to $h'(a')$, $Ind(h'(a')) = Ind(h'(a)) - 1$; hence $Ind(\psi(y)) \leq Ind(\psi(x)) - 1$. A similar argument applied to P'' shows that $Ind(\psi(y))\geq Ind(\psi(x))-1$. Thus $Ind(\psi(y))=Ind(\psi(x))-1$ and $\psi(x)\psi(y)\in E(R)$.

Lemma 2.9. *Assume $x\in K$, $y\in K_2$. If $xy\in E(G)$, then $\psi(x)\psi(y)\in E(R)$.*

Proof. Note that we cannot use our Corollary, as $x\notin L$. However, proceeding by contradiction, since we have $\psi(x)=r_{m+n}$, we may assume that $\psi(y)\leq r_{m+n-2}$. In this case we still may take any $P_y\in\mathcal{P}_y$ and be assured of $l(P_y)\geq 2$, or else $\phi(P_y)\geq r_{m+n-1}$. Thus letting $P''=P_y\circ A^T$, we have $P''\in\mathcal{P}$. There is a homomorphism $P''\to G$ taking $t(P'')=a$ to x. Therefore there is a homomorphism $h'':P''\to R$ such that $h''(a)=r_{m+n}$. Now $h''(i(P''))=r_0$ because r_0 is the only point in C which has positive distance to r_{m+n}. However this is a contradiction because P_y contains no subpath of length n. ∎

Lemma 2.10. *Assume $x \in K_1$, $y \in K_2$. Then $xy \notin E(G)$, and if $yx \in E(G)$, then $l(P_x)=2$ and $l(P_y)=1$ for any $P_x\in\mathcal{P}_x$, $P_y\in\mathcal{P}_y$.*

Proof. Suppose $xy \in E(G)$ and let $P'=P_x\circ A$, for $P_x\in\mathcal{P}_x$. Thus $P'\in\mathcal{P}$ by the Corollary. There is a homomorphism $P' \to G$ taking a to x and a' to y. Hence there is a homomorphism $h':P'\to R$ such that $h'(a')\geq\psi(y)\geq r_{m+1}$. Also, we have $h'(a)\leq\phi(P_x)=\psi(x)\leq r_m$. Hence $h'(a)=r_m$ and $h'(a')=r_{m+n}$. This is impossible, as $r_mr_{m+n}\notin E(R)$.

Now we assume that $yx \in E(G)$. An argument identical to the above (with A replaced by A^T) shows that $h'(a')=r_m$ and $h'(a)=r_{m+n}$. Since $d_R(r_0,r_m)=2$, we have $l(P_x) = d_{P_x}(i(P_x),a') = d_R(h'(i(P_x)),h'(a')) = d_R(r_0,r_m) = 2$, because $h'(i(P_x))=r_0$ and $h'(a')=r_m$.

Let $P_y \in \mathcal{P}_y$; we prove that $l(P_y) = 1$. Let $P'' = P_x \circ A^T \circ (P_y)^T$, and let $f : P'' \to C$ be a homomorphism (it is easy to see that $P'' \to G$, hence also $P'' \to C$). Since $i(P'')$ is the initial point of some minimal interval of P'' of length n, we have $f(i(P'')) = c_0$, and f begins by mapping P'' to C in the positive direction. If $l(P_y) = q \geq 2$, then $l(P'') = 1 - q < 0$, and f must eventually wind P'' around C in the negative direction. But this is impossible, since P_y can not wind around C. Therefore $l(P_y) = 1$. ∎

3. The homomorphism h_ψ

In the previous section we constructed a mapping ψ from G to R, and so, by composition with the natural homomorphism $R \to C$, a mapping from G to C. The above lemmas suggest that ψ is very close to being a homomorphism; however it is not a homomorphism in general. In this section we will modify this mapping to construct a true homomorphism from G to C. Roughly speaking, we shall consider making a correction for those vertices x that are forced into K_1 by a path in $\mathcal{P}_x$ which would allow mapping x further along R if a length zero portion of it were cut out. Specifically:

Definition 3.1. Let M denote the set of all vertices $x \in K_1$ for which there exists a $P_x \in \mathcal{P}_x$, a homomorphism $g : P_x \to G$ with $g(t(P_x)) = x$, and a pair of vertices $u < v \in P_x$ such that $g(u) = g(v)$, $l(P_x[u,v]) = 0$, and $P_x[.,u] \circ P_x[v,.]$ contains no interval of length n. If $x \in M$ then any P_x as above has the same length, and we denote it by $i(x)$.

Define the mapping $h_\psi : G \to V(C)$ as follows:

$$\begin{aligned} h_\psi(x) &= c_0 \quad \text{if} \quad x \in K. \\ h_\psi(x) &= c_i \quad \text{if} \quad x \in K_1 \setminus M \text{ and } \psi(x) = r_i. \\ h_\psi(x) &= c_j \quad \text{if} \quad x \in K_2 \text{ and } \psi(x) = r_{m+n-j}. \\ h_\psi(x) &= c_i \quad \text{if} \quad x \in M \text{ and } i(x) = i. \end{aligned}$$

Theorem 3.2. *The mapping h_ψ is a homomorphism of G to C.*

Proof. Suppose $x, y \in V(G)$ and $xy \in E(G)$. We shall show that $h_\psi(x)h_\psi(y) \in E(C)$. By considering the natural homomorphism $R \to C$ and the definition of h_ψ, the above lemmas imply the following:

If both x and y are in $K_1 \setminus M$ then $h_\psi(x)h_\psi(y) \in E(C)$.

If both x and y are in K_2 then $h_\psi(x)h_\psi(y) \in E(C)$.

If $x \in K$ and $y \in K_2$ then $h_\psi(x)h_\psi(y) \in E(C)$.

We also proved that it is never the case that $x \in K_1$ and $y \in K_2$. Note that it is also never the case that $y \in K$, since r_{m+n} has indegree zero. Therefore we complete the proof of the theorem by showing the following assertions:

If $x \in K$ and $y \in K_1$ then $h_\psi(x)h_\psi(y) \in E(C)$.

If both $x \in M$ and $y \in M$ then $h_\psi(x)h_\psi(y) \in E(C)$.

If $x \in M$, $y \in K_1 \setminus M$ or $y \in M$, $x \in K_1 \setminus M$ then $h_\psi(x)h_\psi(y) \in E(C)$.

If $x \in K_2$ and $y \in K_1$ then $h_\psi(x)h_\psi(y) \in E(C)$.

We proceed to prove these four assertions in a sequence of four lemmas.

Lemma 3.3. *Assume $x \in K$, $y \in K_1$. If $xy \in E(G)$ then $h_\psi(x)h_\psi(y) \in E(C)$.*

Proof. Let $P_y \in \mathcal{P}_y$ and $P = P_y \circ A^T$. Then all vertices $v \in P$ have $0 \le \lambda_P(v) \le n$.

Suppose first that $l(P) > 0$: Then $P \in \mathcal{P}$ and there is a homomorphism $P \to G$ which takes $t(P) = a$ to x. Thus there is a homomorphism $h: P \to R$ such that $h(a) = r_{m+n}$, which implies $h(a') = r_m$ because $h(a') \le \psi(y) \le r_m$. Hence $\psi(y) = r_m$. If $y \notin M$, then $h_\psi(y) = c_m$, which implies $h_\psi(x)h_\psi(y) \in E(C)$. Hence we assume that $y \in M$. We may also assume that P_y fulfills the requirements of the definition of M, i.e., that there exists a homomorphism $g: P_y \to G$ with $g(t(P_y)) = y$, and a pair of vertices $u < v \in P_y$ such that $g(u) = g(v)$, $l(P_y[u,v]) = 0$, and $P' = P_y[.,u] \circ P_y[v,.]$ contains no interval of length n. Let $P'' = P' \circ A^T$. It is clear that $P'' \to R$ taking a to x (since $g(u) = g(v)$ we can use the restriction of g), and that $P'' \in \mathcal{P}$ (because $l(P'') = l(P)$). Since $x \in K$, there is a homomorphism $h'': P'' \to R$ such that $h''(a) = r_{m+n}$. Now the length of P'' is positive, and the only vertex of R with a positive distance to r_{m+n} is r_0. Hence $h''(i(P'')) = r_0$, contrary to P' not containing an interval of length n.

Hence $l(P) = 0$. Then for each vertex $u \in P$, $d_P(i(P),u) = d_P(t(P),u)$; in particular, $l(P_y) = 1$. If $\psi(y) = r_1$, then $h_\psi(y) = c_1$ (whether of not $y \in M$); hence $h_\psi(x)h_\psi(y) \in E(C)$. Thus suppose that $\psi(y) > r_1$. Since $y \in K_1$, there exists in P_y a vertex of level n. Let u be the last such vertex on P_y. Let P^* be a path isomorphic to, but disjoint from, P; let v^* be the vertex of P^* corresponding to the vertex v of P. (For notational reasons, we denote the terminal point by $(a')^* = a''$.) Let $P' = A \circ P_y[u,.]^T \circ P^*_y[u,.]$. Then $P' \to G$ so that the image starts with the edge xy and returns to y. It is easy to verify that $P' \in \mathcal{P}$. Now P' and P_y are two paths in P which admit homomorphisms to G with the terminal points a', a'' taken to y. We claim that $\phi(P') = \phi(P_y)$. It suffices to show that for any homomorphism $h: P' \to R$ there exists a homomorphism $h': P_y \to R$ such that $h'(a') = h(a'')$. Thus let $h: P' \to R$ be a homomorphism. Since $P'[.,u]$ is a minimal interval of length n, we have $h(u) = r_n$. Define $h': P_y \to R$ as follows:

for $v \in P_y[.,u]$ let $h'(v) = r_i$ where $i = \lambda_{P_y}(v)$

for $v \in P_y[u,.]$ let $h'(v) = h(v^*)$.

Then h' is a homomorphism from P_y to R and $h'(a') = h(a'')$. Therefore $\phi(P') = \phi(P_y)$ which implies $P' \in \mathcal{P}_y$. Then the path P' shows that $y \in M$. Since $l(P') = 1$ we have $h_\psi(y) = c_1$ and $h_\psi(x)h_\psi(y) \in E(C)$. ∎

Lemma 3.4. *Assume both $x \in M$ and $y \in M$. If $xy \in E(G)$ then $h_\psi(x)h_\psi(y) \in E(C)$.*

Proof. By an earlier lemma, $\psi(x)\psi(y) \in E(R)$. Let $P_x \in \mathcal{P}_x$, $P_y \in \mathcal{P}_y$, let $h: P_x \to R$ be a homomorphism such that $h(t(P_x)) = \psi(x)$ and $h': P_y \to R$ a homomorphism such that $h'(t(P_y)) = \psi(y)$. Since $h(i(P_x)) = h'(i(P_y)) = r_0$, $l(P_x) = \lambda_R(\psi(x)) = \lambda_R(\psi(y)) - 1 = l(P_y) - 1$. Suppose $l(P_x) = i$ and $l(P_y) = j$. Since $i = j - 1$, $h_\psi(x)h_\psi(y) = c_{j-1}c_j \in E(C)$. ∎

Lemma 3.5. *Assume $x \in M$, $y \in K_1 \setminus M$, or $y \in M$, $x \in K_1 \setminus M$. If $xy \in E(G)$ then $h_\psi(x)h_\psi(y) \in E(C)$.*

Proof. Take paths $P_x \in \mathcal{P}_x$, $P_y \in \mathcal{P}_y$, and let $P' = P_x \circ A \circ P_y^T$. As in lemma 3.4, we find that $l(P_x) = l(P_y) - 1$, i.e., that $l(P') = 0$. Assume first that $x \in M$, $y \in K_1 \setminus M$. Then we may assume that P_x contains vertices $u < v$ such that some homomorphism of P_x to G which takes $t(P_x)$ to x maps u and v to the same vertex of G, and such that $l(P_x[u,v]) = 0$ and $P_x[.,u] \circ P_x[v,.]$ contains no interval of length n. Let also z be the last vertex of P_y of level n. Let $P'' = P_x[.,u] \circ P_x[v,.] \circ A \circ (P_y[z,.])^T \circ P_y[z,.]$. Obviously z is the only vertex of P'' with level n. By the same argument as used in the proof of lemma 3.3, we can show that $P'' \in \mathcal{P}_y$. This would mean that $y \in M$ unless $y = z$. Thus we must have $y = z$ and $\psi(y) = \phi(P'') = r_n$. Therefore $l(P_x) = n-1$ and $h_\psi(x) h_\psi(y) = c_{n-1} c_n \in E(C)$. If $y \in M$, $x \in K_1 \setminus M$, then one finds analogous vertices $u, v \in P_y$, $z \in P_x$, and a corresponding argument applied to $P'' = P_y[.,u] \circ P_y[v,.] \circ A^T \circ (P_x[z,.])^T \circ P_x[z,.]$ shows that $x \in M$, as $l(P_x) \neq n$ by lemma 2.5. Thus this case can not happen, and the lemma is proved. ∎

Lemma 3.6. *Assume $x \in K_2$, $y \in K_1$. If $xy \in E(G)$ then $h_\psi(x) h_\psi(y) \in E(C)$.*

Proof. Let $P_x \in \mathcal{P}_x$, $P_y \in \mathcal{P}_y$. Let u be the last point of P_y with $\lambda_{P_y}(u) = n$. Let $P = P_x \circ A \circ (P_y[u,.])^T \circ P_y[u,.]$. By lemma 2.10, $l(P_x) = 1$, $l(P_y) = 2$. Using an argument from the proof of lemma 3.3 we can show that $P \in \mathcal{P}_y$. If $u = t(P_y)$ then $n = 2$, and $\psi(y) = \phi(P) = r_2$. Hence $h_\psi(x) h_\psi(y) = c_1 c_2 \in E(C)$. If $u \neq t(P_y)$, then P shows that $y \in M$. Again $h_\psi(y) = c_2$ (since $l(P) = 2$) and $h_\psi(x) h_\psi(y) \in E(C)$.

This completes the proof of both of our theorems. ∎

4. Conclusions

For any $\mathcal{B}$-cycle C, our result identifies the obstructions to a possible homomorphism $G \to C$, as oriented paths homomorphic to G but not to C. There are similar obstruction theorems for other graphs and digraphs, [19], [23], [9]. For example, it is well known, [9], [23], [5], that is C is a directed cycle, then $G \to C$ if and only if G contains only cycles of length divisible by the length of C. Perhaps the following may hold:

Conjecture 4.1. *Let C be any oriented cycle of non-zero length. Then $G \to C$ if and only if*

- *each oriented path homomorphic to G is also homomorphic to C and*
- *each oriented cycle of G has length divisible by the length of C.*

Our main theorem verifies the conjecture for $\mathcal{B}$-cycles, which have length 1 and thus automatically satisfy the divisibility condition. The above example verifies the conjecture for directed cycles, which admit a homomorphism from any oriented path and thus automatically satisfy the first condition. We have also verified the conjecture in certain additional cases.

There are corresponding results for oriented paths, [17]:

Theorem 4.2. *Let P be an oriented path. Then $G \to P$ if and only if each oriented path homomorphic to G is also homomorphic to P.*

As mentioned in the introduction, our main motivation in this paper was to prove the following result, [16]:

Theorem 4.3. *Let C be an oriented cycle. Then C is multiplicative if and only if C is a $\mathscr{C}$-cycle.*

We do not wish to define $\mathscr{C}$-cycles here, as the definition is somewhat technical, cf. [16]; the important point is that they are a special class of $\mathscr{B}$-cycles.

Proof-sketch. The necessity of the condition was proved in [26] (cf. also [27]). Thus assume that C is a $\mathscr{C}$-cycle, and $G \not\to C$, $G' \not\to C$. Each $\mathscr{C}$-cycle is a $\mathscr{B}$-cycle, and therefore, according to our main result, there exist paths P, P', homomorphic to G, G' respectively, such that $P \not\to C$ and $P' \not\to C$. We prove in [16] that this implies that there exists a path P^* homomorphic to $P \times P'$ (and hence also homomorphic to $G \times G'$) such that $P^* \not\to C$. Thus we have $G \times G' \not\to C$ and hence C is multiplicative.

As another application of our main result, we shall prove that, for each $\mathscr{B}$-cycle C, the following decision problem is $NP \cap \mathrm{co}NP$:

Instance: A digraph G.

Question: Is G homomorphic to C?

It is easy to see that the problem is in NP. The fact that it also belongs to coNP is an easy consequence (taking $H = C$) of the two lemmas below from [17]. It should be observed that at this time there is no known polynomial algorithm for this problem.

Definition 4.4. Let $P = \langle p_0, p_1, \ldots, p_m \rangle$ be an oriented path and H any digraph. The canonical labeling of P by H is the unique mapping l of P to the subsets of $V(H)$ for which

$l(p_0) = V(H)$

$l(p_{i+1}) = \{v \in V(H)$: for some $u \in l(p_i)$, $uv \in E(H)$ if $p_i p_{i+1} \in E(P)$, or $vu \in E(H)$ if $p_{i+1} p_i \in E(P)\}$.

Lemma 4.5. *Let $P = \langle p_0, p_1, \ldots, p_m \rangle$ be an oriented path and H any digraph. Then $P \to H$ if and only if $l(p_m) \neq \emptyset$ in the canonical labeling of P by H.*

Lemma 4.6. *Let H be a digraph with k vertices, and G a digraph with n vertices. If there exist oriented paths P homomorphic to G but not to H, then there exists such a path P with at most $2^k \cdot n$ vertices.*

Lemma 4.6 implies that any digraph H which admits an obstruction characterization in terms of paths (such as our main theorem, or Theorem 4.2) has a certificate for $G \not\to H$, which is a path of length polynomial in the size of G. (The digraph H is fixed, thus 2^k is a constant.) Then lemma 4.5 verifies the certificate in polynomial time. We noted this for $H = P$, an oriented path, in [17]. These observations extend to any cycle C for which the above conjecture holds, thus the conjecture implies that the existence problem for homomorphism to any oriented cycle is in $NP \cap \mathrm{co}NP$.

Addendum

We have recently obtained a proof of Conjecture 4.1. There is a natural extension of the conjecture, which could include also cycles of length zero, obtained by replacing the second condition with "each oriented cycle of G has length which is a multiple of the length of C". Theorem 4.2 implies that this conjecture holds for the cycles obtained from paths by joining two copies of the same oriented path (identifying the two initial points and the two terminal points). However, we have examples of other zero length cycles which do not satisfy even this version of conjecture. We will discuss this, as well as a new conjecture proposed for all cycles, in a subsequent paper, [18]. As suspected, our proof of the conjecture suggested an idea for a polynomial algorithm for the existence of homomorphisms to cycles of non-zero length, to be presented in [28]. Such an algorithm was independently discovered by W. Gutjahr, who also observed that there exist zero length cycles with NP-complete homomorphism problems. In fact, in a further development, we will show, jointly with J. Nešetřil, that the fact that Conjecture 4.1 holds automatically yields a polynomial algorithm for the homomorphism problems to cycles of non-zero length.

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Pavol Hell, Huishan Zhou
and Xuding Zhu

Simon Fraser University,
Burnaby, B. C,
Canada V5A 1S6
pavol@cs.sfu.ca